ON THE CURVATURE GROUPS OF A CR MANIFOLD

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Dedicated to the memory of Aldo Cossu

Abstract. We show that any contact form whose Fefferman metric admits a nonzero parallel vector field is pseudo-Einstein of constant pseudohermitian scalar curvature. As an application we compute the curvature groups $H^k(C(M),\Gamma)$ of the Fefferman space $C(M)$ of a strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^{n+1}$.

1. Statement of results

Let M be a strictly pseudoconvex CR manifold of CR dimension n and θ a contact form on M such that the Levi form L_θ is positive definite. Let $S^1 \to C(M) \to M$ be the canonical circle bundle and F_θ the Fefferman metric on $C(M)$, cf. [6]. Let $\mathrm{GL}(2n+2,\mathbb{R}) \to L(C(M)) \to C(M)$ be the principal bundle of linear frames tangent to $C(M)$ and $\Gamma : u \in L(C(M)) \mapsto \Gamma_u \subset T_u(L(C(M)))$ the Levi-Civita connection of F_θ. Let $H^k(C(M),\Gamma)$ be the curvature groups of $(C(M),\Gamma)$, cf. [3] and our Section 2. Our main result is

Theorem 1. *If (M,θ) is a pseudo-Einstein manifold of constant pseudohermitian scalar curvature ρ then the curvature groups $H^k(C(M),\Gamma)$ are isomorphic to the de Rham cohomology groups of $C(M)$. Otherwise (that is if either θ is not pseudo-Einstein or ρ is nonconstant) $H^k(C(M),\Gamma)=0$, $1 \le k \le 2n+2$.*

The key ingredient in the proof of Theorem 1 is the explicit calculation of the infinitesimal conformal transformations of the Lorentz manifold $(C(M),F_\theta)$.

Corollary 1. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a smoothly bounded strictly pseudoconvex domain. There is a defining function φ of Ω such that $\theta = \frac{i}{2}(\overline{\partial}-\partial)\varphi$ is a pseudo-Einstein contact form on $\partial\Omega$. If $(\partial\Omega,\theta)$ has constant pseudohermitian scalar curvature then*

$$H^k(C(\partial\Omega),\Gamma) \approx H^k(\partial\Omega,\mathbb{R}) \oplus H^{k-1}(\partial\Omega,\mathbb{R}),$$

for any $1 \le k \le 2n+2$.

The first statement in Corollary 1 is a well known consequence of the fact that $T_{1,0}(\partial\Omega)$ is an embedded CR structure, cf. J.M. Lee, [7]. If for instance Ω is the unit ball in $\mathbb{C}^{n+1}$ and $\theta = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ then

$$H^k(C(\partial\Omega), \Gamma) = \begin{cases} \mathbb{R}, & k \in \{1, 2n+1, 2n+2\}, \\ 0, & \text{otherwise.} \end{cases}$$

The paper is organized as follows. In Section 2 we recall S.I. Goldberg & N.C. Petridis' curvature groups of a torsion-free linear connection (cf. also I. Vaisman, [10]) as well as the needed material on CR manifolds, Tanaka-Webster connection and the Fefferman metric. Section 3 is devoted to the proof of Theorem 1 and corollaries.

2. The curvature groups of the Fefferman metric

Let $(M, T_{1,0}(M))$ be a $(2n+1)$-dimensional connected strictly pseudoconvex CR manifold with the CR structure $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$. Let θ be a contact form on M such that the Levi form

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M),$$

is positive definite. Let $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ be the Levi distribution and

$$J : H(M) \to H(M), \quad J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M),$$

its complex structure. Let $\mathbb{C} \to K(M) \to M$ be the complex line bundle

$$K(M)_x = \{\omega \in \Lambda^{n+1} T^*_x(M) \otimes \mathbb{C} : T_{0,1}(M)_x \rfloor \omega = 0\}, \quad x \in M.$$

There is a natural action of $\mathbb{R}_+$ (the multiplicative positive reals) on $K(M) \setminus \{0\}$ such that $C(M) = (K(M) \setminus \{0\})/\mathbb{R}_+$ is a principal S^1-bundle $\pi : C(M) \to M$ (the canonical circle bundle over M, cf. [2], Chapter 2). The Fefferman metric F_θ is given by

$$F_\theta = \pi^* \tilde{G}_\theta + 2(\pi^*\theta) \odot \sigma, \tag{1}$$

$$\sigma = \frac{1}{n+2}\left\{ d\gamma + \pi^*\left(i\,\omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}} dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\,\theta \right) \right\}. \tag{2}$$

The Fefferman metric is a Lorentz metric on $C(M)$, cf. J.M. Lee, [6]. The following conventions are adopted as to the formulae (1)-(2). Let T be the characteristic direction of $d\theta$ i.e. the tangent vector field on M determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. We set

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

$$\tilde{G}_\theta(X, Y) = G_\theta(X, Y), \quad \tilde{G}_\theta(Z, T) = 0, \quad Z \in T(M).$$

There is a unique linear connection ∇ on M (the Tanaka-Webster connection of (M,θ), cf. [9] and [11]) such that i) the Levi distribution is parallel with respect to ∇, ii) $\nabla J = 0$ and $\nabla g_\theta = 0$, iii) the torsion T_∇ of ∇ is pure i.e.

$$T_\nabla(Z,W) = 0, \quad T_\nabla(Z,\overline{W}) = 2iL_\theta(Z,\overline{W})T, \quad Z,W \in T_{1,0}(M),$$

$$\tau \circ J + J \circ \tau = 0.$$

Here g_θ is the Webster metric i.e. the Riemannian metric on M given by

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1,$$

for any $X,Y \in H(M)$. Also $\tau(X) = T_\nabla(T,X)$, $X \in T(M)$, is the pseudohermitian torsion of ∇. If $\{T_\alpha : 1 \le \alpha \le n\}$ is a local frame of $T_{1,0}(M)$ defined on the open set $U \subseteq M$ then ${\omega_\alpha}^\beta$ are the corresponding connection 1-forms of the Tanaka-Webster connection i.e. $\nabla T_\alpha = {\omega_\alpha}^\beta \otimes T_\beta$. Let R^∇ be the curvature of ∇ and

$$R_{\alpha\overline{\beta}} = \text{trace}\{X \mapsto R^\nabla(X,T_\alpha)T_{\overline{\beta}}\}$$

the pseudohermitian Ricci tensor of (M,θ). Moreover $g_{\alpha\overline{\beta}} = L_\theta(T_\alpha, T_{\overline{\beta}})$ and $\rho = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ is the pseudohermitian scalar curvature of ∇. Also $\gamma : \pi^{-1}(U) \to \mathbb{R}$ is a local fibre coordinate on $C(M)$. Precisely let $\{\theta^\alpha : 1 \le \alpha \le n\}$ be the admissible local coframe associated to $\{T_\alpha : 1 \le \alpha \le n\}$ i.e.

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta\,, \quad \theta^\alpha(T_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0.$$

The locally trivial structure of $S^1 \to C(M) \to M$ is described by

$$\pi^{-1}(U) \to U \times S^1\,, \quad [\omega] \mapsto (x\,,\,\frac{\lambda}{|\lambda|}), \quad \omega \in K(M)_x \setminus \{0\},$$

$$\omega = \lambda(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n)_x\,, \quad x \in U, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Then $\gamma([\omega]) = \arg(\lambda/|\lambda|)$ where $\arg : S^1 \to [0,2\pi)$. If $(U, x^1, \cdots, x^{2n+1})$ is a system of local coordinates on M then $(\pi^{-1}(U), \tilde{x}^1, \cdots, \tilde{x}^m)$ are the naturally induced local coordinates on $C(M)$ i.e. $\tilde{x}^A = x^A \circ \pi$, $1 \le A \le 2n+1$, and $\tilde{x}^m = \gamma$ (with $m = 2n+2$).

Let $\Pi : L(C(M)) \to C(M)$ be the projection and $\rho : \mathrm{GL}(m,\mathbb{R}) \to \mathrm{End}_\mathbb{R}(\mathbb{R}^m)$ the natural representation. We denote by $\Omega^k_{\rho(\mathrm{GL}(m))}(C(M))$ the space of tensorial k-forms of type $\rho(\mathrm{GL}(m,\mathbb{R}))$ i.e. each $\omega \in \Omega^k_{\rho(\mathrm{GL}(m))}(C(M))$ is a $\mathbb{R}^m$-valued k-form on $L(C(M))$ such that

i) $\omega_u(X_1, \cdots, X_k) = 0$ if at least one $X_i \in \mathrm{Ker}(d_u\Pi)$,

ii) $\omega_{ug}((d_uR_g)X_1, \cdots, (d_uR_g)X_k) = \rho(g^{-1})\omega_u(X_1, \cdots, X_k)$ for any $g \in \mathrm{GL}(m,\mathbb{R})$, $X_i \in T_u(L(C(M)))$ and $u \in L(C(M))$.

Let Γ be the Levi-Civita connection of $(C(M), F_\theta)$ thought of as a connection-distribution in $L(C(M)) \to C(M)$. If $\omega \in \Omega^k_{\rho(\mathrm{GL}(m))}(C(M))$ then its covariant derivative with respect to Γ is the tensorial $(k+1)$-form of type $\rho(\mathrm{GL}(m, \mathbb{R}))$

$$(\nabla\omega)(X_0, \cdots, X_k) = \omega(hX_0, \cdots, hX_k),$$

for any $X_i \in T(L(C(M)))$, $0 \le i \le k$. Here $h_u : T_u(L(C(M))) \to \Gamma_u$ is the natural projection associated to the direct sum decomposition $T_u(L(C(M))) = \Gamma_u \oplus \mathrm{Ker}(d_u\Pi)$. Let us consider the $C^\infty(C(M))$-module

$$L^k = \Omega^k_{\rho(\mathrm{GL}(m))}(C(M)) \times \Pi^*\Omega^k(C(M))$$

and the submodule $\tilde{L}^k$ given by

$$\tilde{L}^k = \{(\omega, \Pi^*\alpha) \in L^k : \nabla^2\omega = 0\}.$$

Let $\eta \in \Gamma^\infty(T^*(L(C(M))) \otimes \mathbb{R}^m)$ be the canonical 1-form i.e. $\eta_u = u^{-1} \circ (d_u\Pi)$ for any $u \in L(C(M))$. If we set

$$D^k : \tilde{L}^k \to \tilde{L}^{k+1}, \quad D^k(\omega, \Pi^*\alpha) = (\nabla\omega - \eta \wedge \Pi^*\alpha\,,\, \Pi^* d\alpha),$$

then $\tilde{L} = (\bigoplus_{k=0}^m \tilde{L}^k\,,\, D^k)$ is a cochain complex, cf. [3], p. 550. The curvature groups of Γ are the cohomology groups

$$H^k(C(M), \Gamma) = H^k(\tilde{L}) = \frac{\mathrm{Ker}(D^k)}{D^{k-1}\,\tilde{L}^{k-1}}\,, \quad 1 \le k \le m.$$

3. Infinitesimal conformal transformations

We shall establish the following

Theorem 2. *Any infinitesimal conformal transformation of the Fefferman metric F_θ is a parallel vector field.*

By a result of C.R. Graham, [4], σ is a connection 1-form in $S^1 \to C(M) \to M$. For each vector field $X \in T(M)$ let $X^\uparrow$ denote the horizontal lift of X with respect to σ i.e. $X^\uparrow_z \in \mathrm{Ker}(\sigma_z)$ and $(d_z\pi)X^\uparrow_z = X_{\pi(z)}$ for any $z \in C(M)$. To prove Theorem 2 we need to recall the following

Lemma 1. (E. Barletta et al., [1])
The Levi-Civita connection $\nabla^{C(M)}$ of $(C(M), F_\theta)$ and the Tanaka-Webster connection ∇ of (M, θ) are related by

$$\nabla^{C(M)}_{X^\uparrow} Y^\uparrow = (\nabla_X Y)^\uparrow - (d\theta)(X,Y)T^\uparrow - \{A(X,Y) + (d\sigma)(X^\uparrow, Y^\uparrow)\}S, \tag{3}$$

$$\nabla^{C(M)}_{X^\uparrow} T^\uparrow = (\tau(X) + \phi X)^\uparrow\,, \tag{4}$$

$$\nabla^{C(M)}_{T^\uparrow} X^\uparrow = (\nabla_T X + \phi X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow)S, \tag{5}$$

$$\nabla^{C(M)}_{X^\uparrow} S = \nabla^{C(M)}_{S} X^\uparrow = (JX)^\uparrow\,, \tag{6}$$

$$\nabla^{C(M)}_{T^\uparrow} T^\uparrow = V^\uparrow\,, \quad \nabla^{C(M)}_{S} S = 0, \tag{7}$$

$$\nabla^{C(M)}_{S} T^\uparrow = 0, \quad \nabla^{C(M)}_{T^\uparrow} S = 0, \tag{8}$$

for any $X, Y \in H(M)$. Here $A(X,Y) = g_\theta(\tau(X), Y)$. Also the vector field $V \in H(M)$ and the endomorphism $\phi : H(M) \to H(M)$ are given by

$$G_\theta(V, Y) = 2(d\sigma)(T^\uparrow, Y^\uparrow), \quad G_\theta(\phi X\,,\, Y) = 2(d\sigma)(X^\uparrow, Y^\uparrow),$$

for any $X, Y \in H(M)$.

A vector field $\mathcal{X}$ on $C(M)$ is an infinitesimal conformal transformation of F_θ if

$$\nabla^{C(M)} \mathcal{X} = \lambda I, \tag{9}$$

for some $\lambda \in C^\infty(C(M))$ where I is the identical transformation of $T(C(M))$. Let S be the tangent to the S^1-action (locally $S = \partial/\partial\gamma$). Taking into account the decomposition $T(C(M)) = H(M)^\uparrow \oplus \mathbb{R}T^\uparrow \oplus \mathbb{R}S$ the first order partial differential system (9) is equivalent to

$$\nabla^{C(M)}_{X^\uparrow} \mathcal{X} = \lambda X^\uparrow\,, \;\; \nabla^{C(M)}_{T^\uparrow} \mathcal{X} = \lambda T^\uparrow\,, \;\; \nabla^{C(M)}_{S} \mathcal{X} = \lambda S, \tag{10}$$

for any $X \in H(M)$. Let $\{X_a : 1 \le a \le 2n\} = \{X_\alpha,\ JX_\alpha : 1 \le \alpha \le n\}$ be a local frame of $H(M)$. Then $\mathcal{X} = \mathcal{X}^a X_a^\uparrow + fT^\uparrow + gS$ for some C^∞ functions $\mathcal{X}^a$, f and g. By (6)-(8) in Lemma 1 the last equation in (10) may be written

$$S(\mathcal{X}^a)X_a^\uparrow + \mathcal{X}^a (JX_a)^\uparrow + S(f)T^\uparrow + S(g)S = \lambda S$$

hence

$$\mathcal{X}^a = 0, \quad S(f) = 0, \quad S(g) = \lambda. \tag{11}$$

Similarly, by (4) and (6) in Lemma 1 the first equation in (10) may be written

$$X^\uparrow(f)T^\uparrow + f(\tau(X) + \phi X)^\uparrow + X^\uparrow(g)S + g(JX)^\uparrow = \lambda X^\uparrow$$

hence

$$X^\uparrow(f) = 0, \quad X^\uparrow(g) = 0, \tag{12}$$

$$f(\tau(X)+\phi X)^{\uparrow}+g(JX)^{\uparrow}=\lambda X^{\uparrow}. \tag{13}$$

Lemma 2. *With respect to a local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$ the endomorphism $\phi : H(M)\otimes\mathbb{C} \to H(M)\otimes\mathbb{C}$ is given by $\phi T_\alpha = \phi_\alpha{}^\beta T_\beta + \phi_\alpha{}^{\overline{\beta}} T_{\overline{\beta}}$ with*

$$\phi^{\overline{\alpha}\beta}=\frac{i}{2(n+2)}\{R^{\overline{\alpha}\beta}-\frac{\rho}{2(n+1)}\,g^{\overline{\alpha}\beta}\},\quad \phi^{\alpha\beta}=0, \tag{14}$$

and $\phi^{\overline{\alpha}\beta}=g^{\overline{\alpha}\gamma}\phi_\gamma{}^\beta$ and $\phi^{\alpha\beta}=g^{\alpha\overline{\gamma}}=\phi_{\overline{\gamma}}{}^\beta$.

Proof of Lemma 2. Taking the exterior derivative of (2) we obtain

$$(n+2)d\sigma=\pi^*\left(id\omega_\alpha{}^\alpha-\frac{i}{2}dg^{\alpha\overline{\beta}}\wedge dg_{\alpha\overline{\beta}}-\frac{1}{4(n+1)}\,d(\rho\theta)\right).$$

Note that $\nabla g_\theta=0$ may be locally written as $dg_{\alpha\overline{\beta}}=g_{\alpha\overline{\gamma}}\omega_{\overline{\beta}}{}^{\overline{\gamma}}+\omega_\alpha{}^\gamma g_{\gamma\overline{\beta}}$. Also $g^{\alpha\overline{\beta}}g_{\overline{\beta}\gamma}=\delta^\alpha_\gamma$ yields $dg^{\alpha\overline{\beta}}=-g^{\gamma\overline{\beta}}g^{\alpha\overline{\rho}}dg_{\overline{\rho}\gamma}$. Hence

$$dg^{\alpha\overline{\beta}}\wedge dg_{\alpha\overline{\beta}}=\omega_{\alpha\overline{\beta}}\wedge\omega^{\alpha\overline{\beta}}+\omega_{\overline{\alpha}\beta}\wedge\omega^{\overline{\alpha}\beta}=0.$$

Let $\{\theta^\alpha : 1\le\alpha\le n\}$ be the admissible local coframe associated to $\{T_\alpha : 1\le\alpha\le n\}$. Then (by a result in [11], cf. also [2], Chapter 1)

$$d\omega_\alpha{}^\alpha=R_{\lambda\overline{\mu}}\theta^\lambda\wedge\theta^{\overline{\mu}}+(W^\alpha_{\alpha\lambda}\theta^\lambda-W^\alpha_{\alpha\overline{\mu}}\theta^{\overline{\mu}})\wedge\theta,$$

$$W^\beta_{\alpha\lambda}=g^{\overline{\sigma}\beta}\nabla_{\overline{\sigma}}A_{\alpha\lambda}\,,\quad W^\beta_{\alpha\overline{\mu}}=g^{\overline{\sigma}\beta}\nabla_\alpha A_{\overline{\mu}\,\overline{\sigma}}\,,$$

where $A_{\alpha\beta}=A(T_\alpha,T_\beta)$ and covariant derivatives are meant with respect to the Tanaka-Webster connection. Finally (by the very definition of ϕ)

$$(n+2)G_\theta(\phi X,Y)=i(R_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}})(X,Y)-\frac{\rho}{4(n+1)}(d\theta)(X,Y)$$

for any $X,Y\in H(M)\otimes\mathbb{C}$. This yields (14). Lemma 2 is proved.

Proof of Theorem 2. Let us extend both members of (13) by $\mathbb{C}$-linearity. Then (13) holds for any $X\in H(M)\otimes\mathbb{C}$. By a result in [9] $\tau(T_{1,0}(M))\subseteq T_{0,1}(M)$ hence $\tau(T_\alpha)=A^{\overline{\beta}}_\alpha T_{\overline{\beta}}$ for some C^∞ functions $A^{\overline{\beta}}_\alpha$. Using (13) for $X=T_\alpha$ we obtain

$$f\,A^{\overline{\beta}}_\alpha=0,\quad f\,\phi_\alpha{}^\beta+(ig-\lambda)\delta^\beta_\alpha=0. \tag{15}$$

By Lemma 2

$$\phi_\alpha{}^\beta=\frac{i}{2(n+2)}\left(R_\alpha{}^\beta-\frac{\rho}{2(n+1)}\,\delta^\beta_\alpha\right)$$

and a contraction leads to ${\phi_\alpha}^\alpha = i\rho/[4(n+1)]$. Next a contraction in the second of the identities (15) gives $f\ {\phi_\alpha}^\alpha + n(ig - \lambda) = 0$ or $i\rho f + 4n(n+1)ig = 4n(n+1)\lambda$ and then

$$g = -\frac{\rho}{4n(n+1)}\ f \tag{16}$$

and $\lambda = 0$ as f, g and λ are $\mathbb{R}$-valued. In particular $\nabla^{C(M)}\mathcal{X} = 0$. Theorem 2 is proved.

Corollary 2. *Any infinitesimal conformal transformation of F_θ is a parallel vector field of the form*

$$\mathcal{X} = a\left(T^\uparrow - \frac{\rho}{4n(n+1)}\ S\right), \quad a \in \mathbb{R}. \tag{17}$$

In particular any contact form θ whose Fefferman metric F_θ admits a nontrivial parallel vector field is pseudo-Einstein of constant pseudohermitian scalar curvature and vanishing pseudohermitian torsion.

Proof. By (7)-(8) in Lemma 1 the middle equation in (10) may be written

$$T^\uparrow(f)T^\uparrow + fV^\uparrow + T^\uparrow(g)S = \lambda T^\uparrow$$

hence

$$T^\uparrow(f) = \lambda, \quad T^\uparrow(g) = 0, \tag{18}$$

$$f(z)\ V_{\pi(z)} = 0, \quad z \in C(M). \tag{19}$$

Yet $\lambda = 0$ (by Theorem 2) so that (by (11)-(12) and (18)) $f = a$ and $g = b$ for some $a, b \in \mathbb{R}$. Let us assume now that F_θ admits a parallel vector field $\mathcal{X} \neq 0$. Replacing from (16) into the second of the identities (15) leads to

$$a\left({R_\alpha}^\beta - \frac{\rho}{n}\,\delta_\alpha^\beta\right) = 0. \tag{20}$$

Note that $a \neq 0$ (otherwise (16) implies $b = 0$ hence $\mathcal{X} = 0$) so that (by (20)) $R_{\alpha\overline{\beta}} = (\rho/n)g_{\alpha\overline{\beta}}$ i.e. θ is pseudo-Einstein (cf. [7]). Also (16) shows that $\rho =$ constant. Finally the first identity in (15) implies $\tau = 0$. Corollary 2 is proved.

A remark is in order. Apparently (19) implies that $a = 0$ when $\mathrm{Sing}(V) \neq \emptyset$ (and then there would be no nonzero parallel vector fields on $(C(M), F_\theta)$). Yet we may show that

Corollary 3. *Assume that θ is pseudo-Einstein. Then $V = 0$ if and only if ρ is constant.*

So (19) brings no further restriction. *Proof of Corollary* 3. Note that

$$2(d\omega_\alpha{}^\alpha)(T, T_\beta) = -W^\alpha_{\alpha\beta}\,, \quad 2(d\omega_\alpha{}^\alpha)(T, T_{\overline{\beta}}) = W^\alpha_{\alpha\overline{\beta}}\,,$$

$$2d(\rho\theta)(T, T_\beta) = -\rho_\beta\,, \quad 2d(\rho\theta)(T, T_{\overline{\beta}}) = -\rho_{\overline{\beta}}\,,$$

where $\rho_\beta = T_\beta(\rho)$ and $\rho_{\overline{\beta}} = \overline{\rho_\beta}$. Consequently

$$2(n+2)(d\sigma)(T^\uparrow, T^\uparrow_{\overline{\beta}}) = -iW^\alpha_{\alpha\overline{\beta}} + \frac{1}{4(n+1)}\,\rho_{\overline{\beta}}\,.$$

On the other hand (cf. [2], Chapter 5) if θ is pseudo-Einstein then

$$W^\alpha_{\alpha\beta} = -\frac{i}{2n}\,\rho_\beta\,, \quad W^\alpha_{\alpha\overline{\beta}} = \overline{W^\alpha_{\alpha\beta}}\,.$$

Hence V is given by (see Lemma 1 above)

$$G_\theta(V, T_\beta) = -\frac{1}{4n(n+1)}\,\rho_\beta\,.$$

Clearly if $\rho =$ constant then $V = 0$. Conversely if $V = 0$ then $\overline{\partial}_b\rho = 0$ i.e. ρ is a $\mathbb{R}$-valued CR function. As M is nondegenerate ρ is constant. Corollary 3 is proved.

At this point we may prove Theorem 1. Let $\mathcal{L}^k$ be the sheaf associated to the module $\tilde{L}^k$ i.e. for any open set $A \subseteq C(M)$

$$\mathcal{L}^k(A) = \{(\lambda, \Pi^*\alpha) : \lambda \in \Omega^k_{\rho(\mathrm{GL}(m))}(\Pi^{-1}(A)),\ \ \nabla^2\lambda = 0,\ \ \alpha \in \Omega^k(A)\}.$$

Let $D^k : \mathcal{L}^k \to \mathcal{L}^{k+1}$ be the sheaf homomorphism induced by the module homomorphism $D^k : \tilde{L}^k \to \tilde{L}^{k+1}$.

Lemma 3. *Let $\mathcal{S}_\theta$ be the sheaf of parallel vector fields on $(C(M), F_\theta)$. For each open set $A \subseteq C(M)$ let $j_A : \mathcal{S}_\theta(A) \to \mathcal{L}^0(A)$be given by*

$$j_A(\mathcal{X}) = (f_{\mathcal{X}}, 0), \quad \mathcal{X} \in \mathcal{S}_\theta(A),$$

$$f_{\mathcal{X}} : \Pi^{-1}(A) \to \mathbb{R}^m\,, \quad f_{\mathcal{X}}(u) = u^{-1}(\mathcal{X}_{\Pi(u)}), \quad u \in \Pi^{-1}(A).$$

Then

$$0 \to \mathcal{S}_\theta \xrightarrow{j} \mathcal{L}^0 \xrightarrow{D^0} \mathcal{L}^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{m-1}} \mathcal{L}^m \to 0 \tag{21}$$

is a fine resolution of $\mathcal{S}_\theta$ so that the curvature groups of Γ are isomorphic to the cohomology groups of $C(M)$ with coefficients in $\mathcal{S}_\theta$.

Proof. Let $(f, \lambda) \in \mathcal{L}^0$ such that $0 = D^0(f, \lambda) = (\nabla f - \lambda\eta\,,\, d\lambda)$. Let $z \in C(M)$ and $u \in \Pi^{-1}(z)$. We set by definition $\mathcal{X}_z = u(f(u))$. As $f \circ R_g = \rho(g^{-1}) \circ f$ for any $g \in \mathrm{GL}(m, \mathbb{R})$ it follows that $\mathcal{X}_z$ is well defined. Let $(\Pi^{-1}(C(U)), x^i, X^i_j)$ be the naturally induced local coordinates on $L(C(M))$ i.e. $x^i(u) = \tilde{x}^i(\Pi(u))$ and $X^i_j(u) = a^i_j$ for any $u = (z, \{X_i : 1 \leq i \leq m\}) \in L(C(M))$ such that $X_j = a^i_j(\partial/\partial\tilde{x}^i)_z$

(here $C(U) = \pi^{-1}(U)$). Given a vector field $X = \lambda^j \partial/\partial x^j + \lambda^i_j \partial/\partial X^i_j$ on $L(C(M))$

$$(\nabla f)_u X_u = \lambda^j(u)\, u^{-1}(\nabla^{C(M)}_{\partial/\partial \tilde{x}^j} \mathcal{X})_{\Pi(u)}\,, \quad u \in \Pi^{-1}(C(U)).$$

Then $\nabla f = \lambda \eta$ implies that $\nabla^{C(M)} \mathcal{X} = \lambda I$ hence (by Theorem 2) $\lambda = 0$ i.e. $(f, \lambda) = j(\mathcal{X})$. Therefore the corresponding sequence of stalks $0 \to \mathcal{S}_{\theta,z} \to \mathcal{L}^0_z \to \mathcal{L}^1_z \to \cdots \to \mathcal{L}^m_z \to 0$ is exact at $\mathcal{L}^0_z$ while the exactness at the remaining terms follows from the Poincaré lemma for D as in [3], p. 552. Lemma 3 is proved. In particular if θ is a pseudo-Einstein contact form of constant pseudohermitian scalar curvature then (by Corollary 2) $\mathcal{S}_\theta = \mathbb{R}$ and Lemma 3 furnishes a resolution $0 \to \mathbb{R} \xrightarrow{j} \mathcal{L}^*$ of the constant sheaf $\mathbb{R}$ where

$$j_A(a) = (f_{\mathcal{X}}, 0), \quad a \in \mathbb{R},$$

and $\mathcal{X}$ is given by (17) in Corollary 2, hence

$$H^k(C(M), \Gamma) \approx H^k(C(M), \mathbb{R}), \quad 1 \leq k \leq m.$$

Otherwise (i.e. if θ is not pseudo-Einstein or ρ is nonconstant) then $\mathcal{S}_\theta = 0$. Theorem 1 is proved.

If $M \subset \mathbb{C}^{n+1}$ is a strictly pseudoconvex real hypersurface then (by a result of J.M. Lee, [7]) M admits globally defined pseudo-Einstein contact forms. On the other hand the pullback to M of $dz^0 \wedge \cdots \wedge dz^n$ is a global nonzero section in $K(M)$. In particular $C(M)$ is trivial. If θ is a pseudo-Einstein contact form on M of constant pseudohermitian scalar curvature then (by Theorem 1 and the Künneth formula)

$$H^k(C(M), \Gamma) = H^k(C(M), \mathbb{R}) = \sum_{p+q=k} H^p(M, \mathbb{R}) \otimes H^q(S^1, \mathbb{R}) =$$

$$= H^k(M, \mathbb{R}) \oplus H^{k-1}(M, \mathbb{R})$$

and Corollary 1 is proved. Using M. Rumin's criterion (cf. [8]) for the vanishing of the first Betti number of a pseudohermitian manifold we get

Corollary 4. *Let $M \subset \mathbb{C}^{n+1}$ be a connected strictly pseudoconvex real hypersurface and θ a pseudo-Einstein contact form on M with $\rho =$ constant and $\tau = 0$. If $n \geq 2$ then $H^1(C(M), \Gamma) = \mathbb{R}$.*

An interesting question is whether one may improve Corollary 1 by choosing a contact form with ρ constant to start with. Indeed as $T_{1,0}(M)$ is embedded one may choose a pseudo-Einstein contact form θ. On the other hand if the CR Yamabe invariant $\lambda(M)$ is $< \lambda(S^{2n+1})$ then (by the solution to the CR Yamabe problem due to D. Jerison & J.M. Lee, [5]) there is a positive solution u to the CR Yamabe equation

such that $u^{2/n}\theta$ has constant pseudohermitian scalar curvature. Yet (by a result in [7]) the pseudo-Einstein property is preserved if and only if u is a CR-pluriharmonic function. It is an open problem whether the CR Yamabe equation admits CR-pluriharmonic solutions.

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